Universal $\mathcal{R}$-matrix Of The Super Yangian Double $DY(gl(1|1))$

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Abstract

Based on Drinfel′d realization of super Yangian Double $DY\,(gl(1|1))$, its pairing relations and universal $\mathcal{R}$-matrix are given. By taking evaluation representation of universal $\mathcal{R}$-matrix, another realization $L^{\pm}(u)$ of $DY\,(gl(1|1))$ is obtained. These two realizations of $DY\,(gl(1|1))$ are related by the supersymmetric extension of Ding-Frenkel map.

Yangian algebra was introduced by Drinfel′d[1, 2]. The quantum double of Yangian consists of Yangian itself and its dual with opposite comultiplication. There are three methods to define the Yangian and Yangian double: Drinfel′d-Jmbo [1, 3], Drinfel′d new realization [2] and RS approach [5] (or FRT approach [4] in the case of without center extension). The explicit isomorphism between Drinfel′d new realization and RS realization of Yangian double can be established through Gauss decomposition, the similar method used by Ding and Frenkel in the discussions of quantum Affine algebra. [6]. The property of Yangian double, such as quasi-triangular properties and equivalence of Drinfel′d and RS realization was studied well in some papers [7, 8, 9]. Although the Drinfel′d realization of super Yangian double [10, 11] was constructed by means of RS method and Gauss decomposition, the quasi-triangular property, such as universal $\mathcal{R}$-matrix of super Yangian double has not been studied yet. In this paper, we find the Hopf pairing relations between super Yangian $Y\,(gl(1|1))$ and its dual, then construct the universal $\mathcal{R}$-matrix of $DY\,(gl(1|1))$. By taking evaluation representation, we get the FRT realization of $DY\,(gl(1|1))$.

Super Yangian double $DY\,(gl(1|1))$ is the Hopf algebra generated by elements (Drinfel′d generators) $e_n, f_n, h_n, k_n, n \in \mathbf{Z}$ which satisfy the following multiplication relations

$$[h_m\ ,\ h_n] = [h_m\ ,\ k_n] = [k_m\ ,\ k_n] = 0$$
$$[k_m\ ,\ e_n] = [k_m\ ,\ f_n] = 0$$

$$\begin{array}{l}
[h_0\ ,\ e_n] = -2e_n\ ,\ [k_0\ ,\ f_n] = 2f_n \\
[h_{m+1}\ ,\ e_n] - [h_m\ ,\ e_{n+1}] + \{h_m\ ,\ e_n\} = 0 \\
[h_{m+1}\ ,\ f_n] - [h_m\ ,\ f_{n+1}] - \{h_m\ ,\ f_n\} = 0 \\
\{e_m\ ,\ e_n\} = \{f_m\ ,\ f_n\} = 0 \\
\{e_m\ ,\ f_n\} = -k_{m+n}
\end{array} \tag{1}$$

They could also be written as generating functions (or Drinfel′d currents)

$$E^{\pm}(u) = \pm \sum_{\substack{n\geq 0\\ n<0}} e_n u^{-n-1}\ , \qquad F^{\pm}(u) = \pm \sum_{\substack{n\geq 0\\ n<0}} f_n u^{-n-1} \tag{2}$$

$$H^{\pm}(u) = 1 \pm \sum_{\substack{n\geq 0\\ n<0}} h_n u^{-n-1}\ , \quad K^{\pm}(u) = 1 \pm \sum_{\substack{n\geq 0\\ n<0}} k_n u^{-n-1} \tag{3}$$

and

$$E(u) = E^+(u) - E^-(u)\ , \qquad F(u) = F^+(u) - F^-(u) \tag{4}$$

then the relations (1) look as follows

$$\begin{array}{l}
[H^\sigma(u)\ ,\ H^\rho(v)] = [H^\sigma(u)\ ,\ K^\rho(v)] = [K^\sigma(u)\ ,\ K^\rho(v)] = 0, \qquad \forall \sigma, \rho = +, - \\
[K^{\pm}(u)\ ,\ E(v)] = [K^{\pm}(u)\ ,\ F(v)] = 0 \\
\{E(u)\ ,\ E(v)\} = \{F(u)\ ,\ F(v)\} = 0 \\
H^{\pm}(u)E(v) = \dfrac{u-v-1}{u-v+1} E(v) H^{\pm}(u) \\
H^{\pm}(u)F(v) = \dfrac{u-v+1}{u-v-1} F(v) H^{\pm}(u) \\
\{E(u)\ ,\ F(v)\} = \delta(u-v)[K^-(v) - K^+(u)]
\end{array} \tag{5}$$

in which $\delta(u-v) = \sum_{k\in Z} u^k v^{-k-1}$. The comultiplication structure for $DY\left(gl(1|1)\right)$ is given by

$$\begin{array}{l}
\triangle\left(E^{\pm}(u)\right) = E^{\pm}(u) \otimes 1 + H^{\pm}(u) \otimes E^{\pm}(u) \\
\triangle\left(F^{\pm}(u)\right) = 1 \otimes F^{\pm}(u) + F^{\pm}(u) \otimes H^{\pm}(u) \\
\triangle\left(K^{\pm}(u)\right) = K^{\pm}(u) \otimes K^{\pm}(u) \\
\triangle\left(H^{\pm}(u)\right) = H^{\pm}(u) \otimes H^{\pm}(u) - 2F^{\pm}(u-1)H^{\pm}(u) \otimes H^{\pm}(u)E^{\pm}(u-1))
\end{array} \tag{6}$$

As a quantum double, $DY\left(gl(1|1)\right)$ consists of the super Yangian $Y\left(gl(1|1)\right)$ and its dual $Y^*\left(gl(1|1)\right)$ with opposite comultiplication. The super Yangian $Y\left(gl(1|1)\right)$ is generated by $E^+(u), F^+(u)$,
$H^+(u), K^+(u)$ and $Y^*\left(gl(1|1)\right)$ is generated by $E^-(u), F^-(u), H^-(u), K^-(u)$. There exists a Hopf pairing relation between $Y\left(gl(1|1)\right)$ and $Y^*\left(gl(1|1)\right)$: $<\quad,\quad>$ which satisfies the conditions

$$< ab\ ,\ c^*d^* > = < \triangle(ab)\ ,\ c^* \otimes d^* > = < b \otimes a\ ,\ \triangle(c^*d^*) > \tag{7}$$

for any $a, b \in Y\left(gl(1|1)\right)$ and $c^*, d^* \in Y^*\left(gl(1|1)\right)$. We find that this pairing relation can be written as

$$< E^+(u)\ ,\ F^-(v) > = \frac{1}{u-v}, \qquad < F^+(u)\ ,\ E^-(v) > = \frac{1}{u-v} \tag{8}$$

$$< H^+(u)\ ,\ K^-(v) > = \frac{u-v-1}{u-v+1}, \quad < K^+(u)\ ,\ H^-(v) > = \frac{u-v-1}{u-v+1} \tag{9}$$

As the same discussion for $DY(sl_2)$ [7], the universal $\mathcal{R}$-matrix for $DY\,(gl(1|1))$ has the following form

$$\mathcal{R} = \mathcal{R}_+ \mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_- \tag{10}$$

where

$$\mathcal{R}_+ = \prod_{n\geq 0}^{\rightarrow} \exp(-e_n \otimes f_{-n-1}) \tag{11}$$

$$\mathcal{R}_- = \prod_{n\geq 0}^{\leftarrow} \exp(-f_n \otimes e_{-n-1}) \tag{12}$$

$$\mathcal{R}_1 = \prod_{n\geq 0} \exp\left\{ \mathrm{Res}_{u=v} \left[(-1) \frac{\mathrm{d}}{\mathrm{d}u} (\ln H^+(u)) \otimes \ln K^-(v+2n+1) \right] \right\} \tag{13}$$

$$\mathcal{R}_2 = \prod_{n\geq 0} \exp\left\{ \mathrm{Res}_{u=v} \left[(-1) \frac{\mathrm{d}}{\mathrm{d}u} (\ln K^+(u)) \otimes \ln H^-(v+2n+1) \right] \right\} \tag{14}$$

here we have used the notations

$$\mathrm{Res}_{u=v}\,(A(u) \otimes B(v)) = \sum_k a_k \otimes b_{-k-1} \tag{15}$$

for $A(u) = \sum_k a_k u^{-k-1}$ and $B(u) = \sum_k b_k u^{-k-1}$.

From the quasi-triangular property of the double, the universal $\mathcal{R}$-matrix satisfies

$$\mathcal{R}_{12} \cdot \mathcal{R}_{13} \cdot \mathcal{R}_{23} = \mathcal{R}_{23} \cdot \mathcal{R}_{13} \cdot \mathcal{R}_{12} \tag{16}$$

$$(\triangle \otimes id)\mathcal{R} = \mathcal{R}_{13} \cdot \mathcal{R}_{23}, \qquad (id \otimes \triangle)\mathcal{R} = \mathcal{R}_{13} \cdot \mathcal{R}_{12} \tag{17}$$

In dealing with the tensor product in the graded case, we must use the form $(A \otimes B) \cdot (C \otimes D) = (-1)^{P(B)P(c)} AC \otimes BD$, $P(B) = 0, 1$ for B is bosonic and fermionic respectively.

Let ρ_x be taking two-dimensional evaluation representation for $DY\,(gl(1|1))$:

$$\rho_x(e_n) = \begin{pmatrix} 0 & 0 \\ x^n & 0 \end{pmatrix}, \qquad \rho_x(f_n) = \begin{pmatrix} 0 & x^n \\ 0 & 0 \end{pmatrix} \tag{18}$$

$$\rho_x(h_n) = \begin{pmatrix} x^n & 0 \\ 0 & -x^n \end{pmatrix}, \qquad \rho_x(k_n) = \begin{pmatrix} -x^n & 0 \\ 0 & -x^n \end{pmatrix} \tag{19}$$

and let

$$L^+(x) = (\rho_x \otimes id)(\mathcal{R}^{21})^{-1}, \qquad L^-(x) = (\rho_x \otimes id)\mathcal{R} \tag{20}$$

$$R^+(x-y) = (\rho_x \otimes \rho_y)(\mathcal{R}^{21})^{-1}, \qquad R^-(x-y) = (\rho_x \otimes \rho_y)\mathcal{R} \tag{21}$$

then from (10), we have

$$L^+(x) = \begin{pmatrix} 1 & 0 \\ F^+(x) & 1 \end{pmatrix} \begin{pmatrix} k_1^+(x) & 0 \\ 0 & k_2^+(x) \end{pmatrix} \begin{pmatrix} 1 & E^+(x) \\ 0 & 1 \end{pmatrix} \tag{22}$$

$$L^-(x) = \begin{pmatrix} 1 & 0 \\ F^-(x) & 1 \end{pmatrix} \begin{pmatrix} k_1^-(x) & 0 \\ 0 & k_2^-(x) \end{pmatrix} \begin{pmatrix} 1 & E^-(x) \\ 0 & 1 \end{pmatrix} \tag{23}$$

here

$$k_1^+(x) = \prod_{n\geq 0} \frac{K^+(x-2n-2)}{K^+(x-2n-1)} \frac{H^+(x-2n)}{H^+(x-2n-1)} \tag{24}$$

$$k_2^+(x) = \prod_{n\geq 0} \frac{K^+(x-2n)}{K^+(x-2n-1)} \frac{H^+(x-2n)}{H^+(x-2n-1)} \tag{25}$$

$$k_1^-(x) = \prod_{n\geq 0} \frac{K^+(x+2n+1)}{K^+(x+2n)} \frac{H^+(x+2n+1)}{H^+(x+2n+2)} \tag{26}$$

$$k_2^-(x) = \prod_{n\geq 0} \frac{K^+(x+2n+1)}{K^+(x+2n+2)} \frac{H^+(x+2n+1)}{H^+(x+2n+2)} \tag{27}$$

and

$$R^\pm(x-y) = \rho^\pm(x-y) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{x-y}{x-y+1} & \frac{1}{x-y+1} & 0 \\ 0 & \frac{1}{x-y+1} & \frac{x-y}{x-y+1} & 0 \\ 0 & 0 & 0 & \frac{x-y-1}{x-y+1} \end{pmatrix} \tag{28}$$

here

$$\rho^+(x) = \prod_{n\geq 0} \frac{(x-2n-3)(x-2n-1)^2(x-2n+1)}{(x-2n-2)^2(x-2n)^2} \tag{29}$$

$$\rho^-(x) = \prod_{n\geq 0} \frac{(x+2n)^2(x+2n+2)^2}{(x+2n-1)(x+2n+1)^2(x+2n+3)} \tag{30}$$

From (16), the relations among $R^\pm(x-y)$ and $L^\pm(x)$ can be obtained

$$\begin{aligned} &R_{ij,ab}(x-y)R_{ak,pc}(x-z)R_{bc,qr}(y-z)(-1)^{(P(a)-P(p))P(b)} \\ &\qquad = (-1)^{P(e)(P(f)-P(r))} R_{jk,ef}(y-z)R_{if,dr}(x-z)R_{de,pq}(x-y) \end{aligned} \tag{31}$$

$$\begin{aligned} &R^\pm_{ij,mn}(u-v)L^\pm_{mk}(u)L^\pm_{nl}(v)(-1)^{P(k)(P(n)+P(l))} \\ &\qquad = (-1)^{P(i)(P(j)+P(q))} L^\pm_{jq}(v)L^\pm_{ip}(u)R^\pm_{pq,kl}(u-v) \end{aligned} \tag{32}$$

$$\begin{aligned} &R^-_{ij,mn}(u-v)L^-_{mk}(u)L^+_{nl}(v)(-1)^{P(k)(P(n)+P(l))} \\ &\qquad = (-1)^{P(i)(P(j)+P(q))} L^+_{jq}(v)L^-_{ip}(u)R^-_{pq,kl}(u-v) \end{aligned} \tag{33}$$

The comultiplication structure for current $L^\pm_{ij}(u)$ are got from (17)

$$\triangle\left(L^\pm_{ij}(u)\right) = \sum_{k=1,2} (-1)^{(i+k)(k+j)} L^\pm_{kj}(u) \otimes L^\pm_{ik}(u) \tag{34}$$

The relations (32, 33) and (34) is another defining of super Yangian double, which is usually referred to super version of FRT [4] construction method. If we start from (32, 33) and (34) to define super Yangian double, using the decomposition (22), (23) and setting $K^\pm(u) = k_1^\pm(u)^{-1}k_2^\pm(u), H^\pm(u) = k_1^\pm(u)k_2^\pm(u-1)$, we can also rediscover the Drinfel'd's currents or generators realization of the super Yangian double (1),(5) and (6) [10] .

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